Mathematical Induction

K.DEVENDRAN Msc.,M.Phil., M.Ed.,
Asst Professor Mathematics
SBCEC , Arni.

What is induction?

- A method of proof
- It does not generate answers: it only can prove them
- Three parts:
  - Base case(s): show it is true for one element
  - Inductive hypothesis: assume it is true for any given element
    - **Must be clearly labeled!!!**
  - Show that if it true for the next highest element

Induction example

- Show that the sum of the first *n* odd integers is n^2
  - Example: If $n = 5$, $1+3+5+7+9 = 25 = 5^2$
  - Formally, Show $\forall n \, P(n) \text{ where } P(n) = \sum_{i=1}^{n} 2i - 1 == n^2$
- Base case: Show that P(1) is true

$$P(1) = \sum_{i=1}^{1} 2(i) - 1 == 1^2$$
$$= 1 == 1$$

Induction example, continued

- Inductive hypothesis: assume true for *k*
  - Thus, we assume that P(*k*) is true, or that

$$\sum_{\boxed{i=1}}^{\boxed{k}} 2i-1 == \boxed{k}^2$$

  - Note: we don't yet know if this is true or not!

- Inductive step: show true for *k*+1
  - We want to show that:

$$\sum_{i=1}^{\boxed{k+1}} 2i-1 == \boxed{(k+1)}^2$$

Induction example, continued

- Recall the inductive hypothesis: $\sum_{i=1}^{k} 2i-1 == k^2$

- Proof of inductive step:

$$\sum_{i=1}^{k+1} 2i-1 == (k+1)^2$$

$$2(k+1)-1+\boxed{\sum_{i=1}^{k} 2i-1} == k^2+2k+1$$

$$2(k+1)-1+\boxed{k^2} == k^2+2k+1$$

$$k^2+2k+1 == k^2+2k+1$$

What did we show

- Base case: P(1)
- If P(k) was true, then P(k+1) is true
  - i.e., P(k) → P(k+1)

- We know it's true for P(1)
- Because of P(k) → P(k+1), if it's true for P(1), then it's true for P(2)
- Because of P(k) → P(k+1), if it's true for P(2), then it's true for P(3)
- Because of P(k) → P(k+1), if it's true for P(3), then it's true for P(4)
- Because of P(k) → P(k+1), if it's true for P(4), then it's true for P(5)
- And onwards to infinity

- Thus, it is true for all possible values of n

- In other words, we showed that:

$$\left[\mathrm{P}(1) \wedge \forall k\left(\mathrm{P}(k) \rightarrow \mathrm{P}(k+1)\right)\right] \rightarrow \forall n\, \mathrm{P}(n)$$

The idea behind inductive proofs

- Show the base case
- Show the inductive hypothesis
- Manipulate the inductive step so that you can substitute in part of the inductive hypothesis
- Show the inductive step

Second induction example

- Rosen, section 3.3, question 2:
  - Show the sum of the first n positive even integers is $n^2 + n$
  - Rephrased: $$\forall n\, \mathrm{P}(n) \text{ where } \mathrm{P}(n) = \sum_{i=1}^{n} 2i == n^2 + n$$
- The three parts:
  - Base case
  - Inductive hypothesis
  - Inductive step

Second induction example, continued

- Base case: Show P(1):

$$\mathrm{P}(1) = \sum_{i=1}^{1} 2(i) == 1^2 + 1$$
$$= 2 == 2$$

- Inductive hypothesis: Assume

$$\mathrm{P}(k) = \sum_{i=1}^{k} 2i == k^2 + k$$

- Inductive step: Show

$$\mathrm{P}(k+1) = \sum_{i=1}^{k+1} 2i == (k+1)^2 + (k+1)$$

Second induction example, continued

- Recall our uctive hypothesis: $\mathrm{P}(k) = \sum_{i=1}^{k} 2i == k^2 + k$

$$\sum_{i=1}^{k+1} 2i == (k+1)^2 + k + 1$$

$$2(k+1) + \boxed{\sum_{i=1}^{k} 2i} == (k+1)^2 + k + 1$$

$$2(k+1) + \boxed{k^2 + k} == (k+1)^2 + k + 1$$

$$k^2 + 3k + 2 == k^2 + 3k + 2$$

Notes on proofs by induction

- We manipulate the *k*+1 case to make part of it look like the *k* case
- We then replace that part with the other side of the *k* case

$$\mathrm{P}(k) = \sum_{i=1}^{k} 2i == k^2 + k$$

$$\sum_{i=1}^{k+1} 2i == (k+1)^2 + k + 1$$

$$2(k+1) + \boxed{\sum_{i=1}^{k} 2i} == (k+1)^2 + k + 1$$

$$2(k+1) + \boxed{k^2 + k} == (k+1)^2 + k + 1$$

$$k^2 + 3k + 2 == k^2 + 3k + 2$$

Third induction example

- Rosen, question 7: Show $$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$
- Base case: *n* = 1 $$\sum_{i=1}^{1} i^2 = \frac{1(1+1)(2+1)}{6}$$ $$1^2 = \frac{6}{6}$$ $$1 = 1$$
- Inductive hypothesis: assume $$\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$$

Third induction example

- Inductive step: show $$\sum_{i=1}^{k+1} i^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$$

$$\sum_{i=1}^{k+1} i^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$$

$$(k+1)^2 + \sum_{i=1}^{k} i^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

$$(k+1)^2 + \frac{k(k+1)(2k+1)}{6} = \frac{(k+1)(k+2)(2k+3)}{6}$$

$$6(k+1)^2 + k(k+1)(2k+1) = (k+1)(k+2)(2k+3)$$

$$2k^3 + 9k^2 + 13k + 6 = 2k^3 + 9k^2 + 13k + 6$$

$$\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$$

Third induction again: what if your inductive hypothesis was wrong?

- Show: $$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+\boxed{2})}{6}$$

- Base case: *n* = 1:

$$\sum_{i=1}^{1} i^2 = \frac{1(1+1)(2+2)}{6}$$

$$1^2 = \frac{7}{6}$$

$$1 \neq \frac{7}{6}$$

- But let's continue anyway…
- Inductive hypothesis: assume

$$\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+2)}{6}$$

Third induction again: what if your inductive hypothesis was wrong?

- Inductive step: show $$\sum_{i=1}^{k+1} i^2 = \frac{(k+1)((k+1)+1)(2(k+1)+2)}{6}$$

$$\sum_{i=1}^{k+1} i^2 = \frac{(k+1)((k+1)+1)(2(k+1)+2)}{6}$$

$$(k+1)^2 + \sum_{i=1}^{k} i^2 = \frac{(k+1)(k+2)(2k+4)}{6}$$

$$(k+1)^2 + \frac{k(k+1)(2k+2)}{6} = \frac{(k+1)(k+2)(2k+4)}{6}$$

$$6(k+1)^2 + k(k+1)(2k+2) = (k+1)(k+2)(2k+4)$$

$$2k^3 + 10k^2 + 14k + 6 \neq 2k^3 + 10k^2 + 16k + 8$$

$$\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+2)}{6}$$

Fourth induction example

- Rosen, question 14: show that $n! < n^n$ for all $n > 1$
- Base case: $n = 2$

  $2! < 2^2$

  $2 < 4$
- Inductive hypothesis: assume $k! < k^k$
- Inductive step: show that $(k+1)! < (k+1)^{k+1}$

$$(k+1)! = (k+1)\boxed{k!} < (k+1)\boxed{k^k} < (k+1)(k+1)^k = (k+1)^{k+1}$$

THANK YOU